UDC 532.526

ON SECOND ORDER PARAMETERS WHICH AFFECT THREE-DIMENSIONAL BOUNDARY LAYER SEPARATION

PMM Vol. 38, № 4, 1974, pp. 666-674

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(Received December 12, 1973)

It is shown that the effect of secondary flow on the separation of a three-dimensional boundary layer is determined by the parameters composed of coefficients at terms of second order of smallness in expansions of variables which define the external stream in the neighborhood of particular cross section of the boundary layer. Formulas are derived for appropriate parameters which must be additionally introduced in the separation criterion.

When deriving separation criteria for plane and three-dimensional boundary layers it is customary to use a system of determining parameters which includes coefficients at terms of the first order of smallness in expansions of quantities which determine the external stream in the neighborhood of the separation point. Numerous experiments on the two-dimensional boundary layer show that in a wide class of cases, important from the practical point of view, the neglect of some second order terms is entirely justified. The effect of some second order parameters may become considerable in certain cases of three-dimensional boundary layer separation.

1. Let us consider the number and physical meaning of determining parameters in expressions for the separation criterion, which were derived in [1] and, also, of parameters which specify the stream in the neighborhood of the separation point to within quantities of the second order of smallness.

We locate the origin of a Cartesian system of coordinates at the considered point of a streamlined surface with its x- and y-axes lying in a plane tangent to that surface and the z-axis normal to the latter. We assume for simplicity that the entropy is constant throughout the stream outside the boundary layer and that the motion is vortex-free. We denote quantities at the considered point of the surface by subscript zero. The equations which define the stream outside the boundary layer can now be presented in the form [2]

$$\rho\,(\partial u / \partial x + \partial v / \partial y + \partial w / dz) = -(u\partial\rho / \partial x + v\partial\rho / \partial y + w\partial\rho / \partial z) \tag{1.1}$$

$$\begin{aligned} u\partial u / \partial x + v\partial u / \partial y + w\partial u / \partial z + (1 / \rho)\, \partial p / \partial x &= 0 \\ u\partial v / \partial x + v\partial v / \partial y + w\partial v / \partial z + (1 / \rho)\, \partial p / \partial y &= 0 \\ u\partial w / \partial x + v\partial w / \partial y + w\partial w / \partial z + (1 / \rho)\, \partial p / \partial z &= 0 \end{aligned} \tag{1.2}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0 \tag{1.3}$$

$$p / \rho^\varkappa = p_0 / \rho_0^\varkappa \tag{1.4}$$

To specify in zero approximation the stream at a given point outside the boundary layer it is necessary to set

$$p_0, \ \rho_0, \ u_0, \ v_0 \ (w_0 = 0) \tag{1.5}$$

Phenomena occurring in the boundary layer are affected by the behavior of the outside stream only at the layer boundary. To specify the outside stream in the small neighborhood of a point at the boundary of the boundary layer in the first approximation it is necessary and sufficient to specify derivatives of all quantities with respect to x and y at that point. It follows from (1.4) that the derivatives of density are expressed in terms of pressure derivatives

$$\begin{aligned} (\partial \rho / \partial x)_0 &= (\rho_0 / \varkappa p_0)(\partial p / \partial x)_0 \\ (\partial \rho / \partial y)_0 &= (\rho_0 / \varkappa p_0)(\partial p / \partial y)_0 \end{aligned} \tag{1.6}$$

The first two of Eqs. (1.2) and the first of Eqs. (1.3) yield

$$\begin{aligned} u_0 (\partial u / \partial x)_0 + v_0 (\partial u / \partial y)_0 &= -(\partial p / \partial x)_0 / \rho_0 \\ u_0 (\partial v / \partial x)_0 + v_0 (\partial v / \partial y)_0 &= -(\partial p / \partial y)_0 / \rho_0 \\ (\partial u / \partial y)_0 - (\partial v / \partial x)_0 &= 0 \end{aligned} \tag{1.7}$$

The remaining four equations of system (1.1) – (1.3) contain derivatives $\partial u / \partial z$, $\partial v / \partial z$, and three derivatives of w, i.e. five unknown quantities, hence they cannot be used for determining the derivatives of u and v with respect to x and y. Thus to define the stream in the first approximation at the boundary of a three-dimensional boundary layer it is necessary to specify beside the projection of grad p on the tangential plane (i.e. in addition to $\partial p / \partial x$ and $\partial p / \partial y$) one more combination of derivatives of u and v with respect to x and y. It is logical to specify $(\partial \alpha / \partial n)_0$, as was done in [1], where α is the angle of a streamline to a fixed direction in the tangent plane and n is the distance along the normal to the streamline (also in the tangent plane). Taking α as the angle of the streamline to the x-axis, we obtain

$$\begin{gathered} \operatorname{tg} \alpha = v / u, \quad dx / dn = -\sin \alpha, \quad dy / dn = \cos \alpha \\ \frac{\partial \alpha}{\partial n} = -\sin \alpha \cos^2 \alpha \frac{\partial}{\partial x}\left(\frac{v}{u}\right) + \cos^3 \alpha \frac{\partial}{\partial y}\left(\frac{v}{u}\right) \\ V_0 = (u_0^2 + v_0^2)^{1/2}, \quad \sin \alpha_0 = v_0 / V_0, \quad \cos \alpha_0 = u_0 / V_0 \end{gathered} \tag{1.8}$$

After differentiation and transformation we have

$$\begin{aligned} (\partial \alpha / \partial n)_0 = (1 / V_0^3) \, [u_0^2 (\partial v / \partial y)_0 + v_0^2 (\partial u / \partial x)_0 - \\ u_0 v_0 (\partial v / \partial x + \partial u / \partial y)_0] \end{aligned} \tag{1.9}$$

Solving the system of Eqs. (1.7) and (1.9) for derivatives of velocities, we obtain

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_0 &= \frac{1}{\rho_0 V_0^4}\left[-u_0^2 (V_0^2 + v_0^2)\left(\frac{\partial p}{\partial x}\right)_0 + u_0^2 v_0 \left(\frac{\partial p}{\partial y}\right)_0\right] + \frac{v_0^2}{V_0}\left(\frac{\partial \alpha}{\partial n}\right)_0 \\ \left(\frac{\partial u}{\partial y}\right)_0 = \left(\frac{\partial v}{\partial x}\right)_0 &= -\frac{1}{\rho_0 V_0^4}\left[v_0^3 \left(\frac{\partial p}{\partial x}\right)_0 + u_0^3 \left(\frac{\partial p}{\partial y}\right)_0\right] - \frac{u_0 v_0}{V_0}\left(\frac{\partial \alpha}{\partial n}\right)_0 \\ \left(\frac{\partial v}{\partial y}\right)_0 &= \frac{1}{\rho_0 V_0^4}\left[u_0 v_0^2 \left(\frac{\partial p}{\partial x}\right)_0 - v_0 (V_0^2 + u_0^2)\left(\frac{\partial p}{\partial y}\right)_0\right] + \frac{u_0^2}{V_0}\left(\frac{\partial \alpha}{\partial n}\right)_0 \end{aligned} \tag{1.10}$$

Parameter $(\partial \alpha / \partial n)_0$ defines the convergence or divergence of streamlines in the neighborhood of the considered point. The convergence or divergence can be caused by two phenomena: the three-dimensional character of the stream, and the effect of pressure. With the use of Eqs. (1.7), the equation of continuity (1.1) and formula (1.6), from (1.9) we, in fact, obtain

$$\left(\frac{\partial \alpha}{\partial n}\right)_0 = -\frac{1}{V_0}\left(\frac{\partial w}{\partial z}\right)_0 + (1 - M_0^2)\frac{\pi_0 \cos \varphi_0}{\rho_0 V_0^2} \qquad (1.11)$$

$$\pi_0 = [(\partial p \,/\, \partial x)_0^2 + (\partial p \,/\, \partial y)_0^2]^{1/2} \qquad (1.12)$$

where M_0 is the Mach number at the considered point and φ_0 is the angle between the velocity vector and the projection of the vector $\operatorname{grad} p$ on the tangential plane. The first term in (1.11) depends on the distribution of stream parameters along the normal to the streamlined surface, while the second term depends only on parameters at the surface itself. Thus in the absence of a pressure gradient either divergence or convergence of streamlines may occur at the surface, owing to $(\partial w \,/\, \partial z)_0$ being nonzero (e.g. the flow past a cone at zero angle of attack). On the other hand $(\partial \alpha \,/\, \partial n)_0$ may also be nonzero even in the case of plane external stream (e.g. at flat side walls of a divergent or convergent channel with a straight axis), when $(\partial w \,/\, \partial z)_0 = 0$. The divergence or convergence of streamlines of the external stream leads to the spreading or contraction of the three-dimensional layer, which obviously affects the separation parameter.

This effect corresponds to computations of a three-dimensional boundary layer without allowance for the secondary flow, when the system of equations reduces to a form similar to that of equations for the boundary layer on axisymmetric bodies. The basic difference from the plane case is in the spreading or contraction of the boundary layer with changing radius of the body.

2. The secondary flow can considerably affect the flow in the neighborhood of a wall, i.e. in that region of the boundary layer which has the greatest effect on separation onset. It follows from the foregoing that parameters which define the external stream in the first approximation in the vicinity of the considered boundary layer cross section, obviously, are insufficient for the determination of secondary streams and of possible changes in the criteria of boundary layer separation associated with these. To determine the external stream in the neighborhood of the considered point it is, therefore necessary to resort to the second approximation.

To separate the parameters associated with the secondary flow let us consider the case in which the stream outside the three-dimensional boundary layer is plane, i.e. its properties are independent of z. This occurs, for instance, in the case of flows past cylindrical obstacles standing on a plate. Generally speaking, all statements derived below are valid with reasonable accuracy in the more general case, when properties of the external stream do not strongly depend on z, i.e. when it is possible to neglect in Eqs. (1.1) – (1.3) the derivatives of velocity components with respect to z and take into account only those with respect to the other coordinates. This also applies to the separation region, where derivatives with respect to directions in the tangential plane are usually considerable. When derivatives with respect to z can be neglected, then $(\partial \alpha \,/\, \partial n)_0$ is no longer an independent parameter and, as can be readily derived from (1.11), is expressed in terms of velocity and pressure derivatives

$$(\partial \alpha \,/\, \partial n)_0 = (1 - M_0^2)\,[u_0 (\partial p \,/\, \partial x)_0 + v_0 (\partial p \,/\, \partial y)_0] \,/\, \rho_0 V_0^3 \qquad (2.1)$$

To obtain the parameters which determine in the second approximation the external stream in the neighborhood of a specified point of the surface we differentiate Eqs. (1.1) – (1.4) with respect to x and y (neglecting derivatives with respect to z). From (1.4) we obtain

$$(\partial^2\rho / \partial x^2)_0 = (\rho_0 / \varkappa p_0)\, [(\partial^2 p / \partial x^2)_0 - (\varkappa - 1)\, (\partial p / \partial x)_0^2 / \varkappa p_0] \qquad (2.2)$$

$$(\partial^2\rho / \partial y^2)_0 = (\rho_0 / \varkappa p_0)\, [(\partial^2 p / \partial y^2)_0 - (\varkappa - 1)\, (\partial p / \partial y)_0^2 / \varkappa p_0]$$

$$(\partial^2\rho / \partial x\partial y)_0 = (\rho_0 / \varkappa p_0)\, [(\partial^2 p / \partial x\partial y)_0 - (\varkappa - 1)\, (\partial p / \partial x)_0 (\partial p/\partial y)_0 / \varkappa p_0]$$

These formulas yield a single-valued expression for the second derivatives of density in terms of pressure derivatives. Differentiating with respect to x and y Eq. (1.1), the first two of Eqs. (1.2), and the first of Eqs. (1.3), we obtain seven linearly independent equations (the eighth can be shown to be a combination of these). Taking into account (1.6), we write these equations as

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)_0 = -\frac{1}{\varkappa p_0}\left[\frac{\partial u}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial p}{\partial y} + u\frac{\partial^2 p}{\partial x^2} - \frac{u}{p}\left(\frac{\partial p}{\partial x}\right)^2 + v\frac{\partial^2 p}{\partial x\,\partial y} - \frac{v}{p}\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}\right]_0 \qquad (2.3)$$

$$\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_0 + \left(\frac{\partial^2 v}{\partial y^2}\right)_0 = -\frac{1}{\varkappa p_0}\left[\frac{\partial u}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial p}{\partial y} + u\frac{\partial^2 p}{\partial x\,\partial y} - \frac{u}{p}\frac{\partial p}{\partial x}\frac{\partial p}{\partial y} + v\frac{\partial^2 p}{\partial y^2} - \frac{v}{p}\left(\frac{\partial p}{\partial y}\right)^2\right]_0$$

$$(\partial^2 u / \partial x\partial y)_0 - (\partial^2 v / \partial x^2)_0 = 0, \quad (\partial^2 u / \partial y^2)_0 - (\partial^2 v / \partial x\partial y)_0 = 0$$

$$u_0\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + v_0\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_0 + \left(\frac{\partial u}{\partial x}\right)_0^2 + \left(\frac{\partial u}{\partial y}\right)_0\left(\frac{\partial v}{\partial x}\right)_0 = -\frac{1}{\rho_0}\left(\frac{\partial^2 p}{\partial x^2}\right)_0 + \frac{1}{\varkappa p_0\rho_0}\left(\frac{\partial p}{\partial x}\right)_0^2$$

$$u_0\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_0 + v_0\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)_0 + \left(\frac{\partial u}{\partial x}\right)_0\left(\frac{\partial u}{\partial y}\right)_0 + \left(\frac{\partial v}{\partial x}\right)_0\left(\frac{\partial v}{\partial y}\right)_0 = -\frac{1}{\rho_0}\left(\frac{\partial^2 p}{\partial x\,\partial y}\right)_0 + \frac{1}{\varkappa p_0\rho_0}\left(\frac{\partial p}{\partial x}\right)_0\left(\frac{\partial p}{\partial y}\right)_0$$

$$u_0\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)_0 + v_0\left(\frac{\partial^2 v}{\partial y^2}\right)_0 + \left(\frac{\partial u}{\partial y}\right)_0\left(\frac{\partial v}{\partial x}\right)_0 + \left(\frac{\partial v}{\partial y}\right)_0^2 = -\frac{1}{\rho_0}\left(\frac{\partial^2 p}{\partial y^2}\right)_0 + \frac{1}{\varkappa p_0\rho_0}\left(\frac{\partial p}{\partial y}\right)_0^2$$

Multiplying the first of Eqs. (2.3) by $-u_0$, the second by $-v_0$, and adding these to the fifth and seventh equations, we obtain a relationship which does not contain second derivatives of velocity components. By simple transformations with the use of (1.10), (1.11), and of the equality $(\partial w / \partial z)_0 = 0$, this relationship can be reduced to the form

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_0\left(1 - \frac{\rho_0 u_0}{\varkappa p_0}\right) + \left(\frac{\partial^2 p}{\partial y^2}\right)_0\left(1 - \frac{\rho_0 v_0^2}{\varkappa p_0}\right) - 2\left(\frac{\partial^2 p}{\partial x\,\partial y}\right)_0\frac{\rho_0 u_0 v_0}{\varkappa p_0} = -(\pi_0^2 / \rho_0 V_0^2)\,[2 - 2M_0^2\cos^2\varphi_0 + (1+\varkappa)\, M_0^4\cos^2\varphi_0] \qquad (2.4)$$

The second derivatives of velocity components are uniquely defined by the first six of Eqs. (2.3) in terms of pressure derivatives and zero and first order quantities. The determinant of this system of six linear equations in second derivatives of velocity components is, in fact, $u_0^2 + v_0^2 = V_0^2$, hence it is nonzero at all points at which the absolute value of velocity is nonzero.

Equations (2.2) – (2.4) thus imply that all parameters of the external stream in the neighborhood of the considered point are known, whenever any two independent combinations of second derivatives of pressure are specified. In other words, in the considered case of a plane (or quasiplane) external stream there exist ony two independent second order parameters. The preceding formal derivation does not, however, give any indication as to the physical meaning of these parameters and their relation to the secondary flow in the boundary layer. Let us, therefore, consider in more detail the effect of various factors on the velocity profile in the boundary layer and the formation of the secondary flow.

3. The criterion of separation essentially depends on the magnitude and the pattern of distribution of velocity projections on the direction of the pressure gradient in the boundary layer.

As shown by numerous experiments, the velocity profile in a two-dimensional boundary layer at the separation point is completely determined and thus yields a definite separation criterion. In a three-dimensional boundary layer the velocity profile can vary owing to two effects: first, that of divergence or convergence of streamlines (as already pointed out above) and, second, that of the wall, where the secondary flow can produce a distortion of the velocity profile.

Let us consider possible changes of the velocity profile produced by a flow in the direction normal to the pressure gradient, which we shall henceforth call transverse flow (to avoid any confusion in the terminology, since a flow normal to streamlines of the external flow is usually called secondary flow). The velocity profile in the direction of the pressure gradient will be called basic.

If all the parameters do not vary in the direction normal to the pressure gradient, the velocity of the transverse flow would be constant, and would have no effect on the velocity profile in the direction of the pressure gradient, since outflowing particles would be replaced by the same number of incoming particles with the same parameters. An example of this is the formation of the boundary layer on cylindrical bodies of infinite span subjected to oblique flows.

If the parameters do not remain constant in the direction normal to the pressure gradient, there are two reasons which may result in a substantial change of the basic flow velocity profile.

First, the transverse flow velocity may not be constant when more particles enter a cross section of the boundary layer than are leaving it or vice versa. Since in the region of boundary layer close to the wall the transverse flow exerts a greater effect than the basic flow, hence in the first case the effect is similar to the blowing of gas into that region, while in the second it is that of sucking off gas from it. This may result in the velocity profile in the boundary layer becoming less full in the direction of the pressure gradient in the first case, and more full in the second.

Another effect of the transverse flow may become apparent, when the pattern of velocity profile in the boundary layer considerably varies along the normal to the pressure gradient (e.g. when $\partial\pi / \partial N$ is considerable. $\mathbf{N}$ is a vector in the plane tangential to the surface and orthogonal to the vector of grad p).

Let the transverse flow be in the direction of $\mathbf{N}$ and $\partial\pi / \partial N < 0$. This means that particles carried by the transverse flow come from a region where the basic velocity profile in the boundary layer is less filled than in the considered cross section. Hence particles carried away by the transverse flow are replaced by particles of low kinetic energy, which may lead to an earlier separation of the boundary layer. If, however, $\partial\pi / \partial N > 0$, the incoming particles have a higher kinetic energy and separation may be retarded.

Let us define the parameters which determine the external stream in the neighborhood of the considered point and on which depends the nonuniformity of the transverse flow and of profile fullness in the direction of the pressure gradient. Since no forces are acting along the normal to the pressure gradient, except those of viscosity, it is reasonable to assume that in the first approximation the intensity of the transverse flow is determined by the projection of the external stream velocity vector on the direction of vector $\mathbf{N}$. Along $\mathbf{N}$ this projection may vary owing to the variation of the absolute value of velocity V and of angle φ between vectors $\mathbf{N}$ and grad p. The effect of suction or blowing in the immediate wall neighborhood induced by the transverse flow is the more pronounced the higher is the latter in relation to the basic stream. The relative intensity of suction or blowing may in the first approximation be considered proportional to the variation of the quantity $(\mathbf{V} \cdot \mathbf{N}) / V \cos\varphi = \operatorname{tg}\varphi$ in the direction of $\mathbf{N}$, i.e. to $\partial \operatorname{tg}\varphi / \partial N$. This effect must, also, depend on the profile fullness of the basic flow. When the rate of basic flow at the wall is low (a velocity profile close to separation), a small amount of suction or blowing can considerably affect the profile. If, however, the rate of the basic flow at the wall is high (basic flow with a full velocity profile), the effect of a small amount of suction or blowing will be negligible. Since the fullness of the basic flow velocity profile depends primarily on the pressure gradient (more precisely, on the dimensionless parameter proportional to grad p), it is possible to conclude that the effect of suction or blowing induced by the nonuniformity of the basic flow is defined by the following combination:

$$B = \pi_0 (\partial \operatorname{tg}\varphi / \partial N)_0 \tag{3.1}$$

Variation of the profile fullness along the normal to the pressure gradient depends in the first instance on $\partial\pi / \partial N$, while the effect of that factor on the basic flow is the more pronounced the more intensive the transfer of particles from one cross section to another, i.e. the more intensive the transverse flow. Thus the effect of variation of the basic flow profile along the normal to grad p must, in the first approximation be proportional to the product

$$C = \operatorname{tg}\varphi_0 (\partial\pi / \partial N)_0 \tag{3.2}$$

Let us prove that B and C are independent parameters of the second order. To do this we determine their expressions in terms of pressure derivatives and other hydrodynamic parameters. If $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors along the x- and y-axes, then vector $\mathbf{N}$ is determined by

$$\mathbf{N} = -(\mathbf{i} / \pi)(\partial p / \partial y) + (\mathbf{j} / \pi)(\partial p / \partial x) \tag{3.3}$$

For the scalar product $(\mathbf{V} \cdot \mathbf{N})$, $\cos\varphi$, and the derivative in the direction of $\mathbf{N}$ from (3.3) we obtain formulas (3.4)

$$(\mathbf{V} \cdot \mathbf{N}) = V \sin\varphi = -\frac{u}{\pi}\frac{\partial p}{\partial y} + \frac{v}{\pi}\frac{\partial p}{\partial x}, \qquad \cos\varphi = \frac{u}{\pi V}\frac{\partial p}{\partial x} + \frac{v}{\pi V}\frac{\partial p}{\partial y}$$

$$\partial / \partial N = -(1/\pi)(\partial p/\partial y)(\partial/\partial x) + (1/\pi)(\partial p/\partial x)(\partial/\partial y) \qquad (3.5)$$

After some transformations of (3.1) and (3.3) – (3.5) we obtain

$$B = \frac{\pi_0}{\cos^2\varphi_0}\left(\frac{\partial \varphi}{\partial N}\right)_0 = \frac{1}{\pi_0^2\cos^2\varphi_0}\Big[-\frac{\partial^2 p}{\partial x^2}\left(\frac{\partial p}{\partial y}\right)^2 - \frac{\partial^2 p}{\partial y^2}\left(\frac{\partial p}{\partial x}\right)^2 + 2\frac{\partial^2 p}{\partial x\,\partial y}\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}\Big]_0 + \frac{\pi_0^2\,\mathrm{tg}^2\,\varphi_0}{\rho_0 V_0^2} \qquad (3.6)$$

In a similar manner, from (1.12) and (3.2) – (3.5), we obtain

$$C = \frac{\mathrm{tg}\,\varphi_0}{\pi_0^2}\left\{\left(-\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)\frac{\partial p}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial^2 p}{\partial x\,\partial y}\left[\left(\frac{\partial p}{\partial x}\right)^2 - \left(\frac{\partial p}{\partial y}\right)^2\right]\right\}_0 \qquad (3.7)$$

The determinant D composed of coefficients at second derivatives of pressure in formulas (3.6) and (3.7) for B and C and, also, in formula (2.4), which imposes on the second derivatives of pressure a relationship which must be satisfied for any external flow, is

$$D = (-1 + M_0^2 \cos^2\varphi_0)(\mathrm{tg}\ \varphi_0) / \cos^2 \varphi_0 \qquad (3.8)$$

It follows from (3.8) that $D = 0$, when $\varphi_0 = 0$ or $M_0^2 \cos^2 \varphi_0 = 1$. In the first case the directions of velocity and pressure gradient coincide, and parameter C simply vanishes at the considered point. in the first approximation the transverse flow is absent at the considered point, hence variations of the basic flow profile along the normal to grad p have no effect on the velocity profile at the considered point. In the second case the projection of velocity on the direction of pressure gradient is equal to the speed of sound. Such points in the stream are exceptional. Thus it follows from (3.8) that parameters B and C are independent and not defined in terms of first- and zero-order parameters anywhere, except at singular points.

4. Let us estimate the possible magnitude of the dimensionless parameter formed from B and compare it with the separation parameter of the two-dimensional boundary layer ξ [3]. Since the dimension $[B] = [p] / [L]^2$, where L has the dimension of length, therefore

$$\beta = B z_0^2 / \rho_0 V_0^2 \qquad (4.1)$$

where z_0, a characteristic dimension of the boundary layer, is a dimensionless parameter. For our estimate we consider the flow of an incompressible fluid past a cylinder standing on a plate. Setting the coordinate origin on the plate at the cylinder center and directing the x-and y-axes, respectively, along and perpendicularly to the direction of the oncoming stream, for the absolute value of the stream velocity we obtain the expression [2]

$$V^2 = V_\infty^2\,[1 + 2R^2 / (x^2 + y^2) + R^2\ (R^2 - 4\ x^2) / (x^2 + y^2)^2] \qquad (4.2)$$

where R is the cylinder radius and V_∞ is the absolute value of the oncoming stream velocity. For $y = 0$ at the axis of symmetry the directions of vectors grad p and V coincide, $\cos\varphi_0 = 1$, $\mathrm{tg}\,\varphi_0 = 0$, $(\partial p / \partial y)_0 = 0$, $(\partial p / \partial x)_0 = \pi_0$ and $B = -(\partial^2 p / \partial y^2)_0$, $C = 0$. With the use of the Bernoulli integral and (4.2) we find that at the axis of symmetry

$$B = -2\ \rho V_\infty^2 R^2\ (R^2 - 3\ x^2) / x^6 \qquad (4.3)$$

The flow past a cylinder standing on a plate was experimentally investigated in [4], where it was found that the separation point at the axis of symmetry is at a distance of approximately $^5/_3\,R$ from the cylinder center; from (4.2) and (4.3) we have

$$BR^2 / \rho V^2 \approx 0.7 \qquad (4.4)$$

The length L of the plate from (its) leading edge to the separation point was $L \approx 16\ R$, and the Reynolds number was $Re \approx 4 \cdot 10^6$. The ratio of the turbulent boundary layer displacement thickness in the separation region to the cylinder radius is of the order of 0.1. In that case, taking the displacement thickness δ^* as the characteristic dimension in formula (4.1) and allowing for (4.4), we obtain $\beta \approx 0.018$. Hence the order of magnitude of β is the same as of the criterion of separation of a plane turbulent boundary layer [3]. This confirms that the effects associated with the transverse flow may under certain conditions play an important part in the separation of a three-dimensional boundary layer.

To answer the question whether parameters formed on the basis of higher order derivatives of pressure do play an essential role, we shall estimate parameter $(\partial^4 p / \partial y^4)\ \delta^{*4} / \rho V^2$ (the third derivative $\partial^3 p / \partial y^3$ vanishes at the axis of symmetry). From (4.2) and the Bernoulli integral we find that at the axis of symmetry

$$\partial^4 p / \partial y^4 = \rho V_\infty{}^4 R^2 \,(120 / x^6 - 36\ R^2 / x^8) \qquad (4.5)$$

Setting, as previously, $x = -5R / 3$ and $\delta^* / R = 0.1$, from (4.5) we obtain that in the region of the separation point

$$(\partial^4 p / \partial y^4)\ \delta^{*4} / \rho V^2 \approx 0.001$$

which is by one order of magnitude smaller than β and ξ. It can be readily ascertained that the magnitude of related parameters rapidly decreases with the increasing order of the derivative.

Thus the right hand part of the separation criterion formula derived in [1] must contain in the general case the parameters β and $\gamma = C z_0{}^2 / \rho_0 V_0{}^2$ with some coefficients. The answer to the question how great is the effect of these parameters on separation must await further experimental data, since the available ones are insufficient. Published data (e.g. [4]) make it only possible to conclude that the value of the separation parameter can be substantially greater in the case of three-dimensional separation than in the two-dimensional case.

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Translated by J. J. D.